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STRESS-INDUCED ROTATION OF POLARIZATION DIRECTIONS OF ELASTIC WAVES IN SLIGHTLY ANISOTROPIC MATERIALS

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Abstract—In this paper the effects of the uniformly applied stress and slight anisotropy of materials on the propagation of elastic waves are theoretically studied. Since real polycrystalline materials are more or less anisotropic because of their textures, examining the effects of slight anisotropy is important to establish the method of experimental stress analysis called acoustoelasticity. After deriving the acoustical tensor for the general case of slight anisotropy, the case of slight orthotropy is discussed in detail as a frequently encountered case. The important result is such that the polarization directions of shear waves rotate largely as the uniformly applied stress varies with the principal directions of the stress constant. Such large rotation of the polarization directions does not occur in materials of isotropy or ordinary anisotropy. The effects of this rotation on the acoustical birefringence of two polarized shear waves are also discussed.

1. INTRODUCTION

ELASTIC waves in finitely deformed elastic materials have been studied by many workers. Especially the case of uniform deformations has been discussed thoroughly from the theoretical view point [1–4] and in parallel with the theoretical works the experiments for determining so-called third-order elastic constants of materials by means of the ultrasonic waves have been carried out [5–7].

Furthermore the application of the birefringence of two polarized ultrasonic shear waves to the experimental stress analysis was proposed as acousto–elasticity by Benson and Raelson [8] and has been studied in [9–13]. Though it is attractive that this method involves the possibility of nondestructive three-dimensional stress analysis, there are several problems to be cleared such as the effects of nonuniformity of deformations, some of which were discussed by one of the authors [14], and those of slight anisotropy of materials, which we consider in this paper.

A polycrystalline material is usually considered isotropic when its crystal grains are arranged randomly. However if the grains cluster around certain orientations and so the material has a texture or preferred orientation, it reveals slight anisotropy. Various kinds of textures are caused after plastic deformations, e.g. cold-working and heat treatments, e.g. annealing and most polycrystalline aggregates should be considered to have textures and so more or less possess slight anisotropy intrinsically. Though such slight anisotropy can be neglected in usual elastic deformations, it produces a triad of the polarization directions of elastic waves in the material and the acoustical birefringence of two polarized shear waves is observed by high frequency waves. When such a slightly anisotropic material is loaded, another kind of slight anisotropy is induced by the stress. Since both kinds of anisotropy are weak, the polarization directions rotate largely during the loading which unchanges the principal directions of the stress, for example the loading in a uniaxial tension test.

On the contrary, there is only a little rotation of polarization directions in an ordinary anisotropic material [15] and no rotation of them in an isotropic material [12] during such loading. Due to such large rotation, the relation between the applied stress and the velocities of two polarized shear waves or the phase difference of them becomes a little more complicated than those reported in [12, 15]. Since in two-dimensional acoustoelasticity we intend to determine the stress state from the measurement of the polarization directions and the phase difference of two polarized shear waves, it is necessary to recognize the existence of such rotation of the polarization directions in slightly anisotropic materials to which most polycrystalline aggregates belong.

After deriving the acoustical tensor for the general case of slightly anisotropic materials in Section 2, we confine ourselves to the case of slightly orthotropic symmetry as a frequently encountered case of slight anisotropy and consider the polarization directions, the phase difference and the acoustical birefringence in Sections 3–5.

2. ACOUSTICAL TENSOR FOR SLIGHTLY ANISOTROPIC MATERIALS

In [12] the fundamental equation of the superposed infinitesimal elastic wave was obtained as

$$t_{em}\frac{\partial^2 w_k}{\partial x_l \partial x_m}+\frac{\partial}{\partial x_l}\left[S_{klrs}\cdot\frac{\partial w_r}{\partial x_s}\right] = \rho\ddot{w}_k, \tag{2.1}$$

where

$$S_{klrs} = \frac{\rho}{\rho_0}\cdot\frac{\partial^2\sum}{\partial E_{mn}\partial E_{pq}}\cdot\frac{\partial x_k}{\partial X_m}\cdot\frac{\partial x_l}{\partial X_n}\cdot\frac{\partial x_r}{\partial X_p}\cdot\frac{\partial x_s}{\partial X_q}. \tag{2.2}$$

In equations (2.1) and (2.2) X_k and x_k are coordinates of a material point in the undeformed and the deformed state respectively referred to the same rectangular Cartesian coordinate system and the usual summation convention is used. The deformed state on which the infinitesimal wave w_k is superposed is described by the displacement vector u_k, the strain tensor E_{kl} (referred to the undeformed state) and the stress tensor t_{kl} (referred to the deformed state). Also ρ_0 and ρ are the densities of the undeformed and the deformed state respectively and Σ is the strain energy function. We note that equation (2.1) was derived under the assumption that the second and the higher order terms of the displacement gradient $\partial w_k/\partial X_l$ were negligible. Furthermore we approximate equation (2.1) by neglecting the second and the higher order terms of the displacement gradient $(\partial u_k/\partial X_l)$ $(x_k = u_k(X_l)+X_k)$ as in [14, 15].

Then the strain energy function Σ can be expressed as

$$\sum = \frac{1}{2!}C_{ijkl}E_{ij}E_{kl}+\frac{1}{3!}C_{ijklmn}E_{ij}E_{kl}E_{mn} \tag{2.3}$$

and so we have

$$\frac{\partial^2\sum}{\partial E_{ij}\partial E_{kl}} = C_{ijkl}+C_{ijklmn}E_{mn}. \tag{2.4}$$

where C_{ijkl} and C_{ijklmn} are the second- and the third-order elastic constant. Using equation (2.4), $\partial x_k/\partial X_l = \delta_{kl} + \partial u_k/\partial X_l$ and $\rho = \rho_0(1+E_{kk})$ to equation (2.2) we have

$$S_{klrs} = C_{klrs} + E_{jj}C_{klrs} + \left(\frac{\partial u_k}{\partial X_m}C_{mlrs} + \frac{\partial u_l}{\partial X_m}C_{kmrs} + \frac{\partial u_r}{\partial X_m}C_{klms} + \frac{\partial u_s}{\partial X_m}C_{klrm}\right) + C_{klrsmn}E_{mn}, \quad (2.5)$$

where E_{kl} can be replaced by the linear strain tensor

$$E_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k}\right). \quad (2.6)$$

A slightly anisotropic material is characterized by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + C'_{ijkl} \quad (2.7)$$

and

$$\begin{aligned} C_{ijklmn} = {} & \nu_1\delta_{ij}\delta_{kl}\delta_{mn} + \nu_2[\delta_{ij}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) + \delta_{kl}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{mn}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \\ & + \nu_3[\delta_{ik}(\delta_{jm}\delta_{ln} + \delta_{jn}\delta_{lm}) + \delta_{jl}(\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}) + \delta_{il}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) \\ & + \delta_{jk}(\delta_{im}\delta_{ln} + \delta_{in}\delta_{lm})] + C'_{ijklmn}, \end{aligned} \quad (2.8)$$

where

$$\left|\frac{C_{ijkl}}{\mu}\right| \ll 1, \qquad \left|\frac{C'_{ijklmn}}{\nu_p}\right| \ll 1. \quad (2.9)$$

When $C'_{ijkl} = C'_{ijklmn} = 0$, equations (2.7) and (2.8) represent the elastic constants of an isotropic material [2].

The third-order elastic constants of an isotropic material ν_1, ν_2 and ν_3 are related to the Murnaghan's constants l, m and n [14] by

$$\nu_1 = 2l - 2m + n, \qquad \nu_2 = m - \frac{n}{2}, \qquad \nu_3 = \frac{n}{4}. \quad (2.10)$$

Then taking the condition (2.9) into account, equation (2.5) can be written as

$$S_{klrs} = S^0_{klrs} + S'_{klrs}, \quad (2.11)$$

$$S^0_{klrs} = \lambda\delta_{kl}\delta_{rs} + \mu(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr}), \quad (2.12)$$

$$\begin{aligned} S'_{klrs} = {} & C'_{klrs} + E_{jj}[\lambda\delta_{kl}\delta_{rs} + \mu(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr})] + 2\lambda(\delta_{kl}E_{rs} + \delta_{rs}E_{kl}) \\ & + 2\mu(\delta_{kr}E_{ls} + \delta_{lr}E_{ks} + \delta_{ks}E_{lr} + \delta_{ls}E_{kr}) + \bar{C}_{klrsmn}E_{mn}, \end{aligned} \quad (2.13)$$

where

$$\bar{C}_{klrsmn} = C_{klrsmn} - C'_{klrsmn}. \quad (2.14)$$

If we consider plane waves propagating through a uniformly deformed material, we may assume

$$w_r = W_r\, e^{ik(vt - n_j x_j)}, \quad (2.15)$$

W_r, k, v and n_j being the constant amplitude vector, the wave number, the propagation velocity and the unit vector normal to the wave surface, respectively. Then equation (2.1) becomes

$$(t_{lm}n_l n_m \delta_{kr} + S_{klrs}n_l n_s)W_r = \rho v^2 W_k, \tag{2.16}$$

or

$$[(\lambda E_{jj}\delta_{lm} + 2\mu E_{lm})n_l n_m \delta_{kr} + (S'_{klrs} - E_{jj}S^0_{klrs})n_l n_s + S^0_{klrs}n_l n_s]W_r = \rho_0 v^2 W_k. \tag{2.17}$$

Equation (2.17) is obtained from equation (2.16) by $\rho = \rho_0(1+E_{jj})$, $t_{lm} = \lambda E_{jj}\delta_{lm} + 2\mu E_{lm}$, equation (2.11) and our rule of neglecting the small quantities of higher order. If we put

$$\begin{aligned} A_{kr} &= A^0_{kr} + A'_{kr}, \\ A^0_{kr} &= S^0_{klrs}n_l n_s, \\ A'_{kr} &= (\lambda E_{jj}\delta_{lm} + 2\mu E_{lm})n_l n_m \delta_{kr} + (S'_{klrs} - E_{jj}S^0_{klrs})n_l n_s, \end{aligned} \tag{2.18}$$

equation (2.17) becomes

$$(A_{kr} - \rho_0 v^2 \delta_{kr})W_r = 0. \tag{2.19}$$

The tensor A_{kr} is called the acoustical tensor for the waves with the propagation direction n_j [16] and determines $\rho_0 v^2$ and the polarization directions of that kind of waves as the eigenvalues and the eigenvectors respectively.

Since the case of slight orthotropy is considered in the following sections, we specify C'_{ijkl} for such symmetry here. An orthotropic material is such that its elastic properties are symmetric with respect to three orthogonal planes. So when we choose the coordinate axes to be oriented along the directions of symmetry, C'_{ijkl} is in the Voigt notation [17]

$$\|C'_{ij}\| = \left\| \begin{matrix} C'_{11} & C'_{12} & C'_{31} & & & \\ C'_{12} & C'_{22} & C'_{23} & & 0 & \\ C'_{31} & C'_{23} & C'_{33} & & & \\ & & & C'_{44} & & \\ & 0 & & & C'_{55} & \\ & & & & & C'_{66} \end{matrix} \right\| \tag{2.20}$$

where $|C'_{ij}/\mu| \ll 1$.

3. SLIGHTLY ORTHOTROPIC MATERIALS

Though the eigenvalues and the eigenvectors of A_{kr} can be calculated for the general case by the usual perturbation method [15], we confine ourselves to the case in which one of the directions of orthotropic symmetry and one of the principal directions of the stress are coincident with the propagation direction of the waves. (Though the principal directions of the stress are slightly different from those of the strain on account of the existence of slight anisotropy, this difference can be neglected in our approximation. Therefore we may use the principal directions of the stress and those of the strain interchangeably hereafter.) The above confined case is the one which is frequently encountered in the experiments and reveals the effects of slight anisotropy and the stress on the waves most distinctly.

We choose the coordinate axes so that the x_3-direction is oriented along the above stated common direction and the other two axes along the other two directions of orthotropic symmetry. Then we have

$$n_1 = n_2 = 0, \qquad n_3 = 1, \tag{3.1}$$

$$E_{23} = E_{31} = 0, \qquad E_{33} = E_3, \tag{3.2}$$

and can use the expression (2.20). Using equations (2.8), (2.12)–(2.14), (3.1) and (3.2) to (2.18), we have

$$A^0_{kr} = (\lambda+2\mu)\delta_{k3}\delta_{r3} + \mu\delta_{\alpha\beta}\delta_{k\alpha}\delta_{r\beta}, \tag{3.3}$$

$$\begin{aligned} A'_{kr} = {} & [C'_{3333} + (\lambda+\nu_1+2\nu_2)E_{jj} + \{2(2\lambda+5\mu)+4(\nu_2+2\nu_3)\}E_3]\delta_{k3}\delta_{r3} \\ & + [\{(\lambda+\nu_2)E_{jj} + 2(2\mu+\nu_3)E_3\}\delta_{\alpha\beta} \\ & + (C'_{1313}\delta_{\alpha1}\delta_{\beta1} + C'_{2323}\delta_{\alpha2}\delta_{\beta2}) + 2(\mu+\nu_3)E_{\alpha\beta}]\delta_{k\alpha}\delta_{r\beta}, \end{aligned} \tag{3.4}$$

where Greek indices α and β take the values 1 and 2, and in the Voigt notation $C'_{3333} = C'_{33}$, $C'_{1313} = C'_{55}$ and $C'_{2323} = C'_{44}$. Therefore one of the eigenvalues of A_{kr} is

$$A_3 = (\lambda+2\mu) + [C'_{3333} + (\lambda+\nu_1+2\nu_2)E_{jj} + \{2(2\lambda+5\mu)+4(\nu_2+2\nu_3)\}E_3] \tag{3.5}$$

and the x_3-direction is the corresponding eigendirection. Therefore the velocity v_3 of the longitudinal wave is

$$v_3 = \sqrt{\left(\frac{A_3}{\rho_0}\right)} = V_0\left(1 + \frac{1}{2(\lambda+2\mu)}\{C'_{3333} + (\lambda+\nu_1+2\nu_2)E_{jj} + 2[(2\lambda+5\mu)+2(\nu_2+2\nu_3)]E_3\}\right),$$

where $V_0 = \sqrt{[(\lambda+2\mu)/\rho_0]}$ is the velocity of the longitudinal wave in the undeformed isotropic material. The other two of the eigenvalues and corresponding eigendirections are those of the two-dimensional tensor $A_{\alpha\beta}$:

$$\begin{aligned} A_{\alpha\beta} &= A^0_{\alpha\beta} + A'_{\alpha\beta}, \\ A^0_{\alpha\beta} &= \mu\delta_{\alpha\beta}, \\ A'_{\alpha\beta} &= [(\lambda+\nu_2)E_{jj} + 2(2\mu+\nu_3)E_3]\delta_{\alpha\beta} + C'_{1313}\delta_{\alpha1}\delta_{\beta1} + C'_{2323}\delta_{\alpha2}\delta_{\beta2} + 2(\mu+\nu_3)E_{\alpha\beta}. \end{aligned} \tag{3.6}$$

Due to the existence of the last term in the right hand side of $(3.6)_3$, the x_1- and the x_2-axis are not along the principal directions of $A_{\alpha\beta}$. Since $A_{\alpha\beta}$ is real and symmetrical, its principal directions are orthogonal to each other and its eigenvalues are real. The angle θ between one of the principal directions, that is, polarization directions of the shear waves and the x_1-axis is given by

$$\tan 2\theta = \frac{2A_{12}}{A_{11}-A_{22}} = \frac{2E_{12}}{C'+(E_{11}-E_{22})}, \tag{3.7}$$

where

$$C' = \frac{C'_{1313}-C'_{2323}}{2(\mu+\nu_3)} = \frac{C'_{55}-C'_{44}}{2(\mu+\nu_3)}. \tag{3.8}$$

The characteristic equation which determines the eigenvalues A_α of $A_{\alpha\beta}$ is

$$\tilde{A}^2-[2(\mu+\nu_3)(E_{11}+E_{22})+(C'_{1313}+C'_{2323})]\tilde{A}$$
$$+[2(\mu+\nu_3)E_{11}+C'_{1313}].[2(\mu+\nu_3)E_{22}+C'_{2323}]-4(\mu+\nu_3)^2E_{12}^2 = 0, \tag{3.9}$$

where

$$\tilde{A} = A-[\mu+(\lambda+\nu_2)E_{jj}+2(2\mu+\nu_3)E_3]. \tag{3.10}$$

Therefore we have

$$A_\alpha = \mu+(\lambda+\mu+\nu_2+\nu_3)E_{jj}+(3\mu+\nu_3)E_3$$
$$+\tfrac{1}{2}(C'_{1313}+C'_{2323}) \pm |\mu+\nu_3| . \sqrt{\{[C'+(E_{11}-E_{22})]^2+4E_{12}^2\}}. \tag{3.11}$$

If ϕ is the angle between one of the principal directions of the strain and the x_1-axis, the following relations between $E_{\alpha\beta}$ and the principal strain E_α hold:

$$\begin{aligned} E_{11} &= \tfrac{1}{2}(E_1+E_2)+\tfrac{1}{2}(E_1-E_2)\cos 2\phi, \\ E_{22} &= \tfrac{1}{2}(E_1+E_2)-\tfrac{1}{2}(E_1-E_2)\cos 2\phi, \\ E_{12} &= \tfrac{1}{2}(E_1-E_2)\sin 2\phi. \end{aligned} \tag{3.12}$$

For definiteness, we assume without loss of generality that the eigenvalues A_1 of A_{kl} and E_1 of E_{kl} correspond to the eigendirections $0 \leq \theta < \pi/2$ and $0 \leq \phi < \pi/2$, respectively.

Using the relations (3.12) to equations (3.7) and (3.11) we have

$$\tan 2\theta = \frac{[(E_1-E_2)/C'] . \sin 2\phi}{1+[(E_1-E_2)/C'] . \cos 2\phi}, \tag{3.13}$$

and

$$A_\alpha = \mu+(\lambda+\mu+\nu_2+\nu_3)E_{jj}+(3\mu+\nu_3)E_3$$
$$+\tfrac{1}{2}(C'_{1313}+C'_{2323}) \pm |(\mu+\nu_3) . C'| . \sqrt{\left[1+2\left(\frac{E_1-E_2}{C'}\right)\cos 2\phi+\left(\frac{E_1-E_2}{C'}\right)^2\right]}. \tag{3.14}$$

Since A_α/ρ_0 gives the square of the velocity v_α of the polarized shear wave, the phase difference Φ of the two polarized shear waves/unit length is given by

$$\Phi = \omega\left(\frac{1}{v_2}-\frac{1}{v_1}\right) = \omega\frac{(v_1-v_2)}{v_0^2} = \frac{\omega}{2v_0^3}(v_1^2-v_2^2) = \frac{1}{2\rho_0 v_0^3}(A_1-A_2), \tag{3.15}$$

where ω is the angular frequency of the waves and

$$v_0 = \sqrt{\left(\frac{\mu}{\rho_0}\right)} \quad \left(\frac{|v_\alpha-v_0|}{v_0} \ll 1\right).$$

Therefore we have

$$\Phi = \pm\left(\frac{\omega}{v_0}\right) . \left|\left(1+\frac{\nu_3}{\mu}\right) C'\right| . \sqrt{\left[1+2\left(\frac{E_1-E_2}{C'}\right) . \cos 2\phi+\left(\frac{E_1-E_2}{C'}\right)^2\right]}, \tag{3.16}$$

where the positive or negative sign corresponds whether $A_1 > A_2$ or $A_1 < A_2$.

Hereafter we suppose that the material is loaded so that the principal directions of the strain do not change as in a usual uniaxial tension test. If the principal directions of the strain are coincident with the directions of orthotropic symmetry, that is $\phi = 0$, we have from relations (3.13) and (3.16)

$$\theta = 0,$$
$$\Phi = \pm\left(\frac{\omega}{v_0}\right).\left|\left(1+\frac{\nu_3}{\mu}\right).C'.\left\{1+\left(\frac{E_1-E_2}{C'}\right)\right\}\right|, \tag{3.17}$$

which shows that the polarization directions are coincident with the principal directions of the strain during the loading and the phase difference is the simple sum of the contributions from the intrinsic orthotropy and the stress-induced orthotropy.

If the material is isotropic, i.e. $C' = 0$, then we have

$$\theta = \phi$$
$$\Phi = \pm\left(\frac{\omega}{v_0}\right).\left|\left(1+\frac{\nu_3}{\mu}\right).(E_1-E_2)\right| \tag{3.18}$$
$$= \pm\left(\frac{\omega}{v_0\mu}\right).\left|\left(1+\frac{\nu_3}{\mu}\right).(t_1-t_2)\right|,$$

where t_α is the principal stress. Relation (3.18) is called the stress–acoustical law. The polarization directions are also constant during the loading (ϕ = const.).

The general behaviors of the rotation of the polarization directions and the change of the phase difference are shown in Figs. 1 and 2. The polarization directions rotate largely

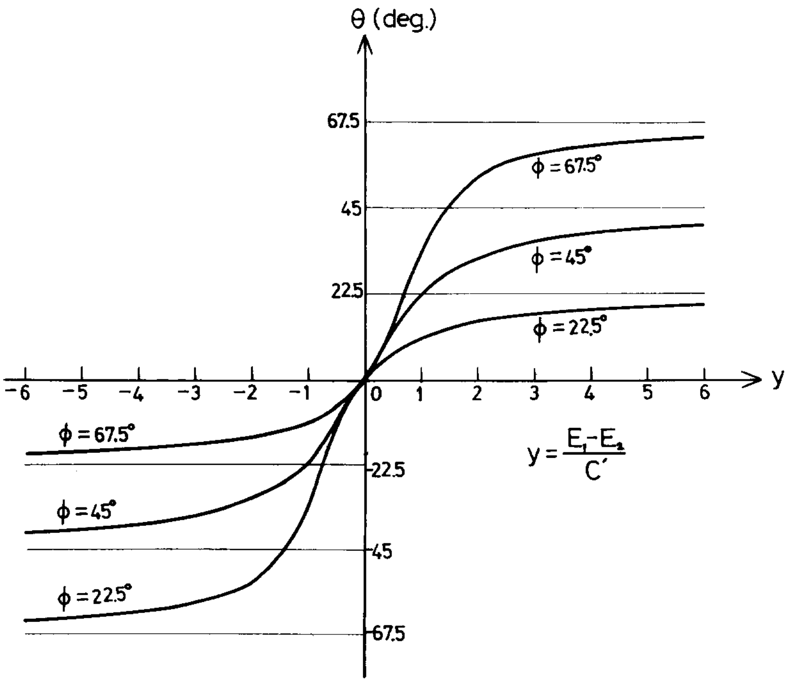

FIG. 1. Variation of θ with $y \equiv (E_1 - E_2)/C'$. In this figure θ is the angle between the x_1-axis and the polarization direction which is coincident with the x_1-axis when $y = 0$. Since this definition of θ makes easier to see the figure than that in the article, i.e. $0 \leq \theta \leq \pi/2$, the former is adopted here exceptionally.

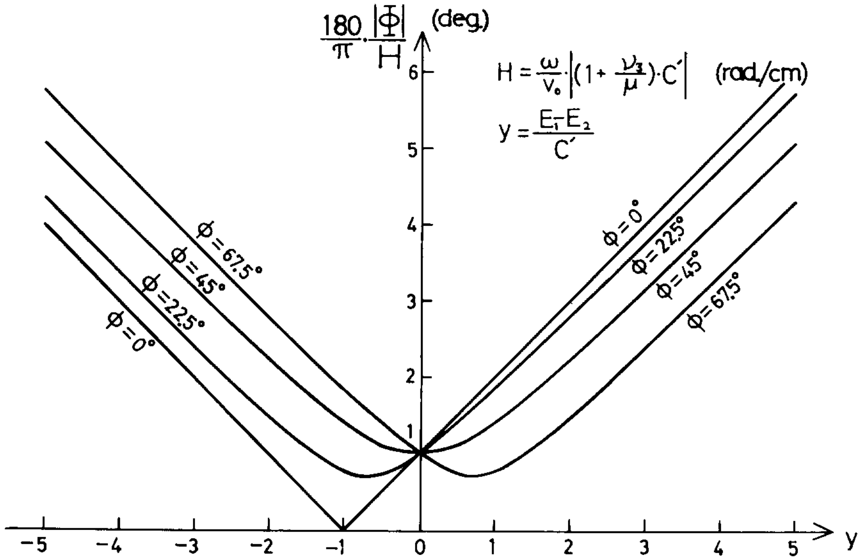

FIG. 2. Variation of Φ with $y \equiv (E_1 - E_2)/C'$.

from those of orthotropic symmetry $[(E_1 - E_2)/C' = 0]$ towards those of the principal axes of the stress $[|(E_1 - E_2)/C'| \gg 1]$ during the loading [that is, with the increase of $|(E_1 - E_2)/C'|$]. Since within the elastic range $|(E_1 - E_2)/C'|$ does not exceed the value 1 extremely, the latter directions can not be reached. Such large rotation of the polarization directions as this is due to slightness of orthotropy and so it does not occur in the materials of ordinary anisotropy. Even if these materials are stressed, the polarization directions deviate only a little from those determined by the intrinsic anisotropy. Also in the isotropic materials they do not deviate at all as shown in equation (3.17).

4. THE AMPLITUDE OF A SHEAR WAVE-ECHO PATTERNS

The case is the same as that treated in Section 3, that is, the coordinate axes are oriented along the directions of orthotropic symmetry and the x_3-axis is also along one of the principal directions of the stress. When on the plane $x_3 = 0$ the uniform displacement vector w_k is given by

$$\begin{aligned} w_1 &= W.\cos\psi\,.\cos\omega t, \\ w_2 &= W.\sin\psi\,.\cos\omega t, \\ w_3 &= 0, \end{aligned} \tag{4.1}$$

where W is its magnitude and ψ is the angle between this vector and the x_1-axis, the plane shear waves propagate into the positive and the negative direction of the x_3-axis. We consider one of them, the shear wave propagating into the positive x_3-direction. Then its displacement at a point x_3 along the direction making the angle ψ with the x_1-axis is given by

$$w(x_3, t) = w_1'(x_3, t)\,.\cos(\psi - \theta) + w_2'(x_3, t)\,.\sin(\psi - \theta), \tag{4.2}$$

where θ is also the angle between one of the polarization directions and the x_1-axis and w'_1 and w'_2 are components of the displacement vector along the polarization directions (Fig. 3). Since as shown in Section 3 the components w'_1 and w'_2 are two waves with the velocities v_1 and v_2, respectively, we have

$$\begin{aligned} w'_1(x_3, t) &= W.\cos(\psi-\theta).\cos\left[\omega.\left(t-\frac{x_3}{v_1}\right)\right], \\ w'_2(x_3, t) &= W.\sin(\psi-\theta).\cos\left[\omega.\left(t-\frac{x_3}{v_2}\right)\right]. \end{aligned} \tag{4.3}$$

v_1 and v_2 are given from equation (3.14). From equations (4.2) and (4.3)

$$w(x_3, t) = W\left\{\cos^2(\psi-\theta).\cos\left[\omega\left(t-\frac{x_3}{v_1}\right)\right]+\sin^2(\psi-\theta).\cos\left[\omega\left(t-\frac{x_3}{v_2}\right)\right]\right\}. \tag{4.4}$$

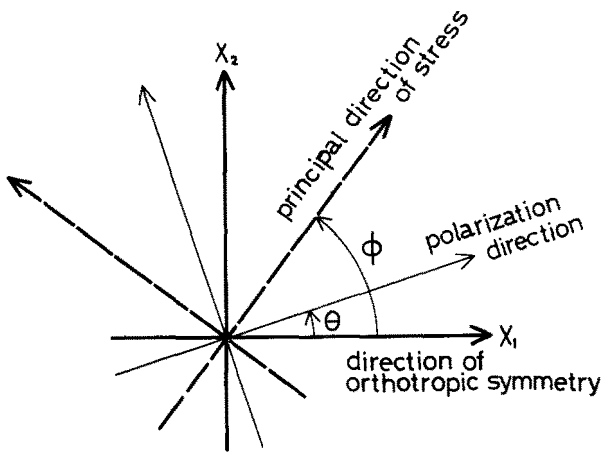

FIG. 3. Three triads of directions.

When we write equation (4.4) as

$$w(x_3, t) = W.r.\cos(\omega t-\zeta), \tag{4.5}$$

we have

$$\begin{aligned} r = r(x_3) &= [1-\sin^2 2(\psi-\theta).\sin^2(\tfrac{1}{2}\Phi.x_3)]^{\frac{1}{2}}, \\ \zeta = \zeta(x_3) &= \tan^{-1}\left[\frac{\tan^2(\psi-\theta).\sin(\Phi.x_3)}{1+\tan^2(\psi-\theta).\cos(\Phi.x_3)}\right] \end{aligned} \tag{4.6}$$

using the phase difference Φ/unit length given by equation (3.15) and so equation (3.16). $W.r(x_3) \equiv w(x_3)_{max}$ is the amplitude of the displacement $w(x_3, t)$ along the direction ψ.

To proceed to the explanation of formula (4.6) we briefly refer to what kind of experiments gives the above derived amplitude $w(x_3)_{max}$. Consider a uniaxial tension or compression test in which the test piece is a slightly orthotropic plate with a thickness d and moreover the direction of the thickness is along one of the directions of orthotropic symmetry

(x_3-axis). In order to generate a shear wave pulse with the main frequency component $\omega/2\pi$ and detect its reflected pulses, we set a Y-cut quartz crystal on the plane $x_3 = 0$ so that its direction of vibration makes the angle ψ with the x_1-axis (Fig. 4). Then for a certain load we obtain a pattern of successively reflected wave pulses on the screen of the oscilloscope. This pattern is called the echo pattern and is such as Fig. 5. The observed height of the n-th reflected pulse is proportional to the amplitude of the displacement along the

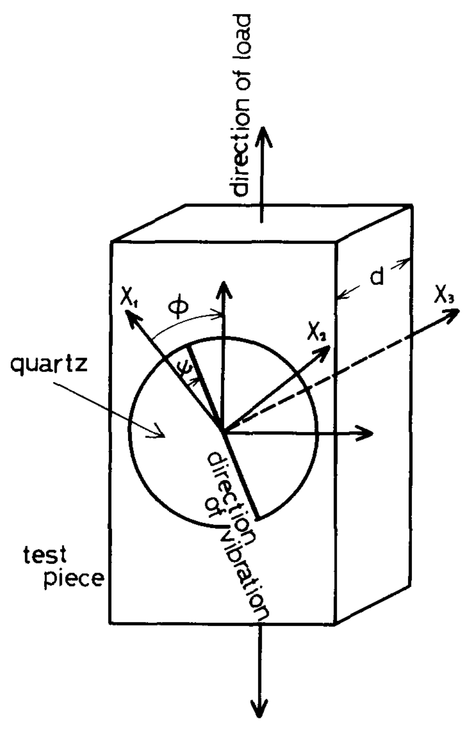

FIG. 4. The acoustoelastic experiment under uniaxial stress.

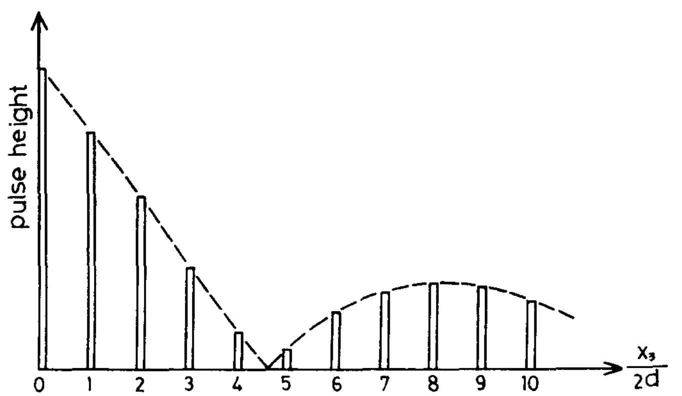

FIG. 5. An echo pattern in acoustoelasticity.

direction ψ at $x_3 = 2nd$ $(n = 0, 1, 2, \ldots)$. When we draw the curve connecting the tops of the observed pulses (the dotted line in Fig. 5) and the proportionality factor is adequately chosen, this curve gives the relation between r and x_3. (In reality the loss in reflecting on the surfaces of the specimen and that due to scattering on the grain boundaries etc. decrease the echo pulse heights. However we do not consider these losses in this paper, so the cause of the change of the echo pulse heights is simply the acoustical birefringence [18].)

Returning to formula (4.6), we first consider the case $\phi = 0$ (the principal directions of the stress are coincident with the directions of orthotropic symmetry). Then formula (4.6) becomes

$$r = \left|\cos\left[\frac{\omega x_3}{2v_0}\cdot\left(1+\frac{v_3}{\mu}\right).C'.\left(1+\frac{E_1-E_2}{C'}\right)\right]\right| \tag{4.7}$$

by using equation (3.17) and putting $\psi = \pi/4$. The change of the echo pattern (the r–x_3 curve) with the increase of the value $|(E_1-E_2)/C'|$ is shown in Figs. 6(a) and (b) for $(E_1-E_2)/C' \geq 0$ and $(E_1-E_2)/C' \leq 0$, respectively. In Fig. 6(a) the phase difference increases monotonically with $(E_1-E_2)/C'$ and so the period of the curve becomes shorter and shorter, while in Fig. 6(b) the period increases infinitely and thenceforth decreases as in Fig. 6(a). The echo pattern with an infinite period is the straight line with a height 1 and shows that in this stress state the intrinsic slight orthotropy is counterbalanced by the stress-induced orthotropy and the material is two-dimensionally isotropic. Of course, this is made possible by the choice $\phi = 0$. Another kind of curve which gives the relation

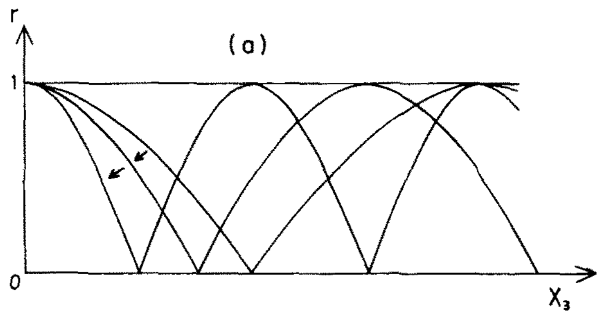

FIG. 6(a). Change of the echo pattern for $\phi = 0°$, $\psi = 45°$ and $y \equiv (E_1-E_2)/C' > 0$ as y increases.

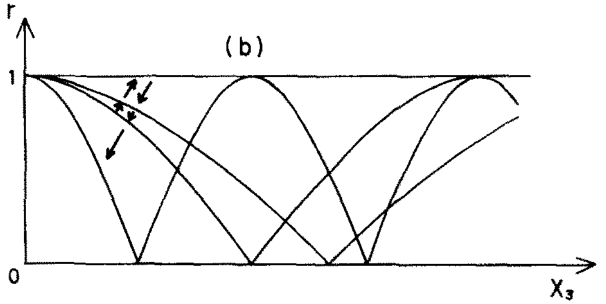

FIG. 6(b). Change of the echo pattern for $\phi = 0°$, $\psi = 45°$ and $y \equiv (E_1-E_2)/C' < 0$ as $|y|$ increases.

between r and $(E_1-E_2)/C'$ at a certain point x_3 is important to analyse the stress. It is shown in Fig. 7. Its period is inversely proportional to x_3.

Secondly for the case $C'=0$ and $\psi=\pi/4$ we show the echo pattern and the $r-(E_1-E_2)$ curve in Figs. 8 and 9, respectively. In this case we have

$$r = \left|\cos\left[\frac{\omega x_3}{2v_0}\cdot\left(1+\frac{v_3}{\mu}\right).(E_1-E_2)\right]\right|. \tag{4.8}$$

Interpretation of these figures is similar to the above.

Finally for the general case we have to apply equation (3.16) to equation (4.6). Putting $\psi=\phi=\pi/4$ for brevity, we have

$$r = \left(1-\cos^2 2\theta\,.\sin^2\left\{\frac{\omega x_3}{2v_0}\cdot\left(1+\frac{v_3}{\mu}\right).C'.\sqrt{\left[1+\left(\frac{E_1-E_2}{C'}\right)^2\right]}\right\}\right)^{\frac{1}{2}}. \tag{4.9}$$

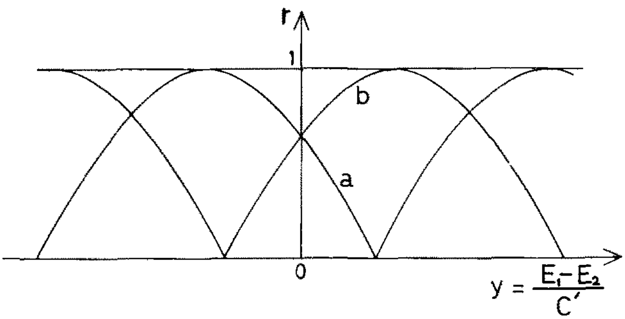

FIG. 7. Variation of r with $y \equiv (E_1-E_2)/C'$ when $\phi = 0°$ and $\psi = 45°$. When

$$p\pi < \frac{\omega x_3}{2v_0}\left|\left(1+\frac{|v_3|}{\mu}\right).C'\right| < (p+\tfrac{1}{2})\pi \qquad (p=0,1,2,\ldots),$$

curves a and b correspond to $v_3>0$ and $v_3<0$, respectively. Also when

$$(p+\tfrac{1}{2})\pi < \frac{\omega x_3}{2v_0}\cdot\left|\left(1+\frac{|v_3|}{\mu}\right).C'\right| < (p+1)\pi \qquad (p=0,1,2,\ldots),$$

curves a and b correspond to $v_3<0$ and $v_3>0$, respectively.

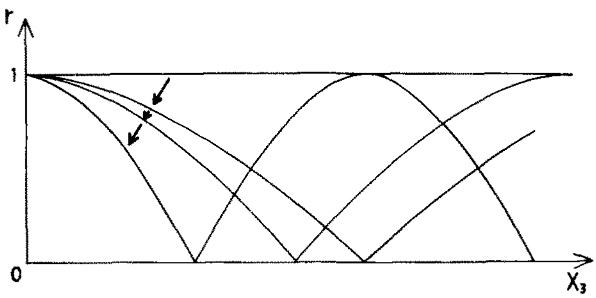

FIG. 8. Change of the echo pattern for $C'=0$ and $\psi=45°$ as $|y| \equiv |(E_1-E_2)/C'|$ increases.

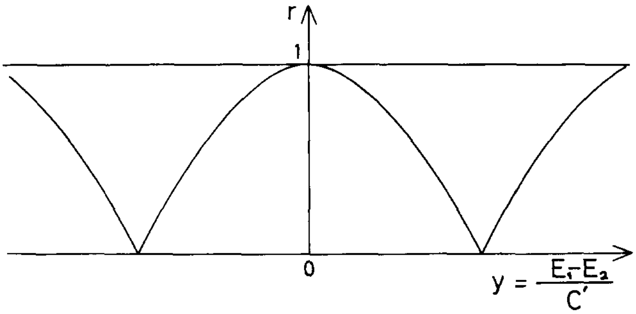

FIG. 9. Variation of r with $y \equiv (E_1 - E_2)/C'$ when $C' = 0$ and $\psi = 45°$.

Since $\cos^2 2\theta$ is expressed as

$$\cos^2 2\theta = \frac{1}{1+\tan^2 2\theta} = \frac{1}{1+[(E_1-E_2)/C']^2} \tag{4.10}$$

by equation (3.13), relation (4.9) becomes

$$r = \left[1 - \frac{\sin^2(\omega x_3/2v_0 \,.\, (1+\nu_3/\mu) \,.\, C' \,.\, \sqrt{\{1+[(E_1-E_2)/C']^2\}})}{1+[(E_1-E_2)/C']^2}\right]^{\frac{1}{2}}. \tag{4.11}$$

The echo pattern and the $r-[(E_1-E_2)/C']$ curve are shown in Figs. 10 and 11. They differ from and are more complicated than the curves in two special cases shown above. Therefore it can be concluded that the effects of large rotation of the polarization directions due to the applied stress are very important in slightly anisotropic materials.

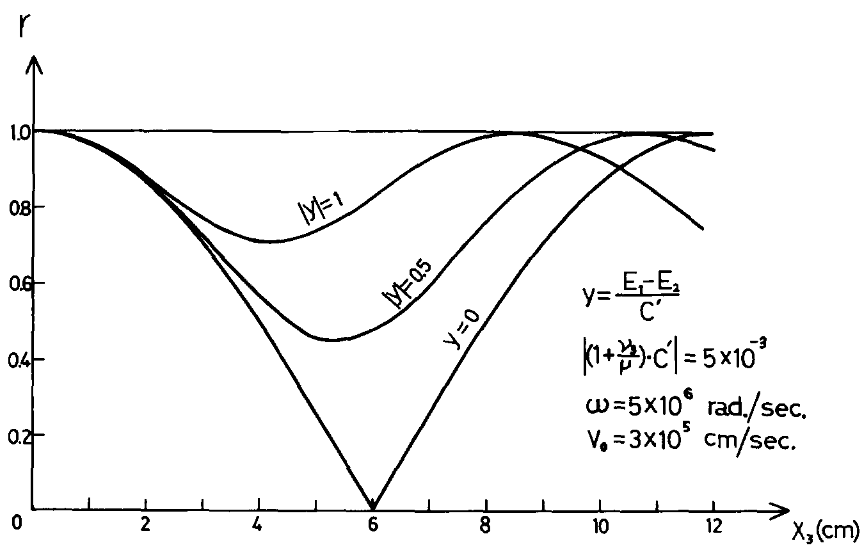

FIG. 10. Change of the echo pattern with $|y| \equiv |(E_1 - E_2)/C'|$ when $\phi = \psi = 45°$.

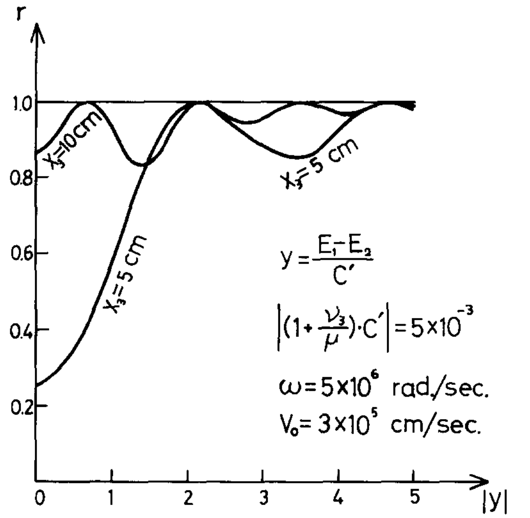

FIG. 11. Variation of r with $|y| \equiv |(E_1-E_2)/C'|$ when $\phi = \psi = 45°$.

5. DISCUSSION

As an example of slightly orthotropic materials we refer to a rolled plate of a metal. It has a texture in which most grains arrange with certain crystallographical axes parallel to the direction of rolling, and this direction and that orthogonal to it in the plane of the plate are two directions of orthotropic symmetry. This kind of orthotropy was studied in [13, 19, 20] for the rolled plates of iron and aluminium by using the ultrasonic shear waves and from those results we may suppose that $(\Delta v)_0$, the difference of the velocities of the two shear waves polarized by this orthotropy, is at most about 1 per cent of the velocity $v_0 = \sqrt{(\mu/\rho_0)}$ of the shear wave in the corresponding isotropic material. [Of course, the ratio $(\Delta v)_0/v_0$ depends on the properties of the material and its reduction in thickness in rolling.] Then taking $(\Delta v)_0/v_0 = 5\times10^{-3}$ as a representative ratio and assuming the case in relation (3.17), we compare the phase differences Φ_i due to the intrinsic orthotropy and Φ_s due to the applied stress. From (3.17) Φ_i and Φ_s are

$$\Phi = \pm|\Phi_i+\Phi_s|,$$

$$\Phi_i = \frac{\omega}{v_0}\cdot\left(1+\frac{\nu_3}{\mu}\right).C' = \frac{\omega}{v_0}\cdot\left(\frac{C'_{55}-C'_{44}}{2\mu}\right), \tag{5.1}$$

$$\Phi_s = \frac{\omega}{v_0}\cdot\left(1+\frac{\nu_3}{\mu}\right).(E_1-E_2).$$

Since

$$\Phi_i = \frac{\omega}{v_0}\cdot\frac{(\Delta v)_0}{v_0},$$

we have

$$|\Phi_i| = 5\times10^{-3}\,.\left(\frac{\omega}{v_0}\right) \quad \text{and} \quad \left|\frac{C'_{55}-C'_{44}}{2\mu}\right| = 5\times10^{-3} \tag{5.2}$$

$(5.2)_2$ suggests

$$\left|\frac{C'_{55}}{2\mu}\right|, \quad \left|\frac{C'_{44}}{2\mu}\right| \ll 1, \tag{5.3}$$

though this cannot be strictly obtained from (5.2). So we may assume that the condition of slightness of orthotropy (2.9) is satisfied for these rolled plates.

Though for the polycrystalline iron and aluminium the authors have not known the definite experimental values of the third-order elastic constant $\nu_3 = n/4$, Seeger and Buck [21] gave $n = -15{\cdot}2\times10^{12}$ dyn/cm^2 for iron. Adopting this value for n and assuming the magnitude of the strain as $|E_1-E_2| = 10^{-3}$, we have

$$|\Phi_s| \doteqdot 3{\cdot}7\times10^{-3}\,.\left(\frac{\omega}{v_0}\right). \tag{5.4}$$

From $(5.2)_1$ and (5.4) it proves that the intrinsic slight orthotropy and the stress-induced slight orthotropy equally contribute to the phase difference Φ. Though this comparison is made in the case when the directions of orthotropic symmetry are along the principal directions of the stress and so the polarization directions do not rotate, the same is concluded for the general case, in which we cannot divide Φ into Φ_i and Φ_s as shown in (3.16). Therefore it is essential in interpreting the experimental data to recognize that the polarization directions rotate largely during the loading.

Finally it should be noted that the residual stress affects the propagation of the elastic waves as well as the intrinsic slight anisotropy and the applied stress, and so these three kinds of factors have to be taken into account in applying the method of acoustoelasticity to the real materials [13].

REFERENCES

[1] D. S. Hughes and J. L. Kelly, *Phys. Rev.* **92**, 1145 (1953).
[2] R. A. Toupin and B. Bernstein, *J. acoust. Soc. Am.* **33**, 216 (1961).
[3] M. Hayes and R. S. Rivlin, *Archs ration. Mech. Analysis* **8**, 15 (1961).
[4] R. N. Thurston and K. Brugger, *Phys. Rev.* **133**, 1604 (1964).
[5] T. B. Bateman, W. P. Mason and H. J. McSkimin, *J. appl. Phys.* **32**, 928 (1961).
[6] H. J. McSkimin and P. Andreatch, Jr., *J. appl. Phys.* **35**, 3312 (1964).
[7] E. H. Bogardus, *J. appl. Phys.* **36**, 2504 (1964).
[8] R. W. Benson and V. J. Raelson, *Product Engng* **30**, 56 (1956).
[9] F. R. Rollins, WADC Technical Report 59-561 (1959).
[10] R. W. Benson, *Ultrasonic News*, p. 14 (1962).
[11] E. E. Day and R. A. Brewer, *JSME* 1967 *Semi-International Symposium*, Tokyo, p. 147 (1967).
[12] T. Tokuoka and Y. Iwashimizu, *Int. J. Solids Struct.* **4**, 383 (1968).
[13] H. Lee and Y. Torikai, *Seisan-Kenkyu* **21**, 379 (1969).
[14] Y. Iwashimizu, *Int. J. Solids Struct.* **7**, 419 (1971).
[15] T. Tokuoka and M. Saito, *J. acoust. Soc. Am.* **45**, 1241 (1969).
[16] C. Truesdell, *Archs ration. Mech. Analysis* **8**, 263 (1961).
[17] R. N. Thurston, *Physical Acoustics*, edited by Warren P. Mason, Vol. 1, Part A, chapter 1. Academic Press (1964).

[18] P. C. WATERMAN and L. J. TEUTONICO, *J. appl. Phys.* **28**, 266 (1957).
[19] F. A. FIRESTONE and J. R. FREDERICK, *J. acoust. Soc. Am.* **18**, 200 (1946).
[20] R. F. SULLIVAN and E. P. PAPADAKIS, *J. acoust. Soc. Am.* **33**, 1622 (1961).
[21] A. SEEGER and O. BUCK, *Z. Naturf.* **15a**, 1056 (1960).

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Абстракт—В работе исследуются теоретически эффекты равномерно приложенных напряжений и легкой анизотропии материалов на распространение упругих волн. Пока действительные поликристаллические материалы более или менее анизотропные, в виду их структуры, тогда является важным-исследование эффектов легкой анизотропии, с цельы определения метода экспериментального анализа напряжений, названного акусто-упругостью. После вывода акустического тензора для общего случая легкой анизотропии, исследуется подробно случай легкой ортотропии, в смысле часто встречающего случая. Самым важным результатом является то, что направления поляризации волн сдвига обращаются более, чем изменяется равномерно приложенное напряжение, в зависимости от главных направлений постоянной напряжения. Так большое вращение направлений поляризации не встречается в изотропных материалах, или с обыкновенной анизотропией. Исследуется, также, эффект этого вращения на акустическое двойное лучепреломление двух поляризированных волн сдвига.